A CONTINUOUS FREQUENCY DOMAIN DESCRIPTION OF ADJUSTABLE BOUNDARY CONDITIONS FOR MULTIDIMENSIONAL TRANSFER FUNCTION MODELS

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ABSTRACT

Physical modeling of string vibrations strongly depends on the conditions at the system boundaries. The more complex the boundary conditions are the more complex is the process of physical modeling. Based on prior works, this contribution derives a general concept for the incorporation of complex boundary conditions into a transfer function model designed with simple boundary conditions. The concept is related to control theory and separates the treatment of the boundary conditions from the design of the string model.

1. INTRODUCTION

The physical modeling of vibrating strings, e.g. guitar or piano strings is a well studied topic in the area of sound synthesis. A description of the physical phenomena in terms of partial differential equations (PDE) allows the application of different modeling techniques. These methods transform the physical description into computational models by transformations into the respective frequency domain or by discretization in time and space [1, 2].

While a PDE describes the vibrations of a guitar string itself, a suitable set of boundary conditions has to be chosen to achieve a realistic scenario. For a guitar string different kinds of boundary conditions are discussed in [3]. Simple boundary conditions can be defined as conditions for deflection (fixed ends) or its first order space derivatives (free ends). Complex boundary conditions, e.g. impedance boundary conditions are described by a linear combination of physical quantities by a complex boundary impedance.

A suitable modelling technique is the Functional Transformation Method (FTM), based on an expansion into eigenfunctions [4]. These eigenfunctions are most easily determined for simple boundary conditions [5]. The incorporation of complex boundary conditions leads to non-linear equations for the eigenvalues [6].

An alternative approach for the incorporation of complex boundary conditions is based on a suitable state-space description. It is shown for the FTM in [7] for the case of frequency-independent impedance boundary conditions. These concepts are highly related to the basics of open-loop and closed loop systems in control theory as the synthesis algorithm is kept separate from the boundary model [8–10]. Thus the eigenvalues of the simple boundary value problem act as eigenvalues of the open loop system and the eigenvalues of the complex boundary value problem belong to the closed loop system. The closed loop eigenvalues do not need to be calculated explicitly, instead their effect is created by the feedback loop.

This contribution extends the concepts presented in [7, 11] in the continuous frequency domain which hold for a wide range of PDE's of physical interest. The synthesis models based on the FTM are formulated in terms of a state-space description and a well defined input/output model for the system boundaries is derived. This model leads to a generalized boundary circuit, which shows how the simple boundary conditions are connected to more complex ones.

The presentation is not completely self-contained, since important steps have already been discussed in [7] and references therein. Therefore readers are occasionally referred to the corresponding sections of [7] for details not presented here.

The paper is structured as follows: Sec. 2 gives an overview on the Functional Transformation Method to obtain a transfer function model by modal expansion. Subsequently the synthesis of such systems is reformulated in terms of a state space description in Sec. 3. Sec. 4 presents an extensive input/output model for physical systems and clarifies the terms simple and complex boundary conditions. Especially the connection between both kinds of boundary conditions is highlighted in this section, which leads to a modified state space model in Sec. 5. Sec. 6 recapitulates the string model presented in [7], which is reformulated to fit into the input/output model. In Sec. 7 a bridge model for a guitar is proposed, which is connected to the string model. The simulation results are shown in Sec. 8. Finally Sec. 9 presents several ways in which the model can be extended and applied in further works.

2. MULTIDIMENSIONAL TRANSFER FUNCTION MODEL

For the synthesis of physical systems a multidimensional transfer function model can be obtained by suitable transformation methods, e.g. with the Functional Transformation Method. The method is based on a combination of the Laplace transformation for the time variable and an expansion into spatial eigenfunctions by the Sturm-Liouville transformation.

2.1. Physical Description

The PDEs describing a physical system along with their initial and boundary conditions can be formulated in vector form. The application of a Laplace transform removes the time derivatives and leaves a set of ordinary differential equations. Linear systems which are at rest for $t < 0$ can be formulated in a generic vector form on the spatial region $0 \leq x \leq \ell$

$$[s\boldsymbol{C} - L]\, \boldsymbol{Y}(x,s) = \boldsymbol{F}_\mathrm{e}(x,s), \qquad L = \boldsymbol{A} + \boldsymbol{I}\frac{\partial}{\partial x}, \quad (1)$$

with the spatial differential operator L and the complex frequency variable s. The vector $\boldsymbol{F}_\mathrm{e}(x,s)$ is the continuous frequency domain equivalent of a vector of time and space dependent functions

$\boldsymbol{f}_\mathrm{e}(x,t)$ exciting the system. The matrix $\boldsymbol{C}$ contains weighting parameters for the time derivatives, and the matrix $\boldsymbol{A}$ contains loss parameters and is included in the spatial differential operator L. The vector of variables $\boldsymbol{Y}(x,s)$ contains the unknown physical quantities. The matrix $\boldsymbol{I}$ is the identity matrix.

The boundary conditions of the system prescribe the existence of a boundary value $\boldsymbol{\Phi}$ for one variable or a linear combination of variables in the vector $\boldsymbol{Y}(x,s)$ at $x=0,\ell$. The conditions are formulated in terms of the vector of boundary values and a matrix of boundary conditions $\boldsymbol{F}_\mathrm{b}^\mathrm{H}$ acting on the vector of variables

$$\boldsymbol{F}_\mathrm{b}^\mathrm{H}(x,s)\boldsymbol{Y}(x,s) = \boldsymbol{\Phi}(x,s), \qquad x=0,\ell, \tag{2}$$

where the superscript H denotes the hermitian matrix (conjugate transpose).

2.2. Sturm-Liouville Transformation

For application to the space dependent quantities in (1) a Sturm-Liouville Transformation (SLT) is defined in terms of an integral transformation [4]

$$\bar{Y}(\mu,s) = \mathcal{T}\{\boldsymbol{Y}(x,s)\} = \int_0^\ell \tilde{\boldsymbol{K}}^\mathrm{H}(x,\mu)\boldsymbol{C}\boldsymbol{Y}(x,s)\mathrm{d}x, \tag{3}$$

where the vector $\tilde{\boldsymbol{K}}$ is the kernel of the transformation and $\bar{Y}(\mu,s)$ is the representation of the vector $\boldsymbol{Y}(x,s)$ in the complex temporal and spatial transform domain. Defining a suitable kernel $\boldsymbol{K}$, the inverse transformation is expressed in terms of a generalized Fourier series expansion

$$\boldsymbol{Y}(x,s) = \mathcal{T}\{\bar{Y}(\mu,s)\} = \sum_{\mu=-\infty}^{\infty} \frac{1}{N_\mu}\bar{Y}(\mu,s)\boldsymbol{K}(x,\mu), \tag{4}$$

with the scaling factor N_μ. The kernel functions are unknown in general, they can be determined for each problem (1) as the solution of an arising eigenvalue problem [4,12]. The integer index μ is the index of a discrete spatial frequency variable s_μ for which the PDE (1) has nontrivial solutions [4].

2.3. Properties of the Transformation

The SL-transform described above has to fulfill different properties to be applicable to a PDE (1), e.g. the discrete nature of the eigenvalues s_μ, the adjoint nature of the kernel functions $\boldsymbol{K}$ and $\tilde{\boldsymbol{K}}$ and the bi-orthogonality of the kernel functions. The properties of the SLT and the FTM are described in detail in [4, 12]. The mathematical derivations regarding the SLT and the FTM are omitted here for brevity, as this contribution is focussed on the input and output behaviour at the system boundaries.

2.4. Transform Domain Representation

The application of the transformation (3) turns a PDE in the form of Eq. (1) into an algebraic equation of the form (see [7, Eq. (16)])

$$s\bar{Y}(\mu,s) - s_\mu\bar{Y}(\mu,s) = \bar{F}_\mathrm{e}(\mu,s) + \bar{\Phi}(\mu,s), \tag{5}$$

with the transformed vector of boundary values and the transformed excitation function

$$\bar{\Phi}(\mu,s) = -\left[\tilde{\boldsymbol{K}}^\mathrm{H}(x,\mu)\boldsymbol{\Phi}(x,s)\right]\Big|_0^\ell, \tag{6}$$

$$\bar{F}_\mathrm{e}(\mu,s) = \int_0^\ell \tilde{\boldsymbol{K}}^\mathrm{H}(x,\mu)\boldsymbol{F}_\mathrm{e}(x,s)\mathrm{d}x. \tag{7}$$

Solving (5) for the transformed vector of variables leads to the representation

$$\bar{Y}(\mu,s) = \bar{H}(\mu,s)\left[\bar{F}_\mathrm{e}(\mu,s) + \bar{\Phi}(\mu,s)\right], \tag{8}$$

with the multidimensional transfer function

$$\bar{H}(\mu,s) = \frac{1}{s-s_\mu}, \qquad \mathrm{Re}\{s_\mu\} < 0. \tag{9}$$

3. STATE SPACE MODEL

The description by a multidimensional transfer function from Sec. 2 can be formulated in terms of a state-space description, basically shown in [7]. This formulation provides many computational benefits (e.g. to avoid delay free loops) and allows the incorporation of boundary conditions of differnt kinds.

3.1. State Equation

The transform domain representation (5) is reformulated to a form which resembles a state equation for each single value of μ

$$s\bar{Y}(\mu,s) = s_\mu\bar{Y}(\mu,s) + \bar{F}_\mathrm{e}(\mu,s) + \bar{\Phi}(\mu,s). \tag{10}$$

Combining all μ-dependent variables into matrices and vectors, expands the state equation to any number of eigenvalues s_μ

$$s\bar{\boldsymbol{Y}}(s) = \boldsymbol{\mathcal{A}}\bar{\boldsymbol{Y}}(s) + \bar{\boldsymbol{\Phi}}(s) + \bar{\boldsymbol{F}}_\mathrm{e}(s), \tag{11}$$

with the vectors

$$\bar{\boldsymbol{Y}}(s) = \left[\cdots,\quad \bar{Y}(\mu,s),\quad \cdots\right]^\mathrm{T}, \tag{12}$$

$$\bar{\boldsymbol{\Phi}}(s) = \left[\cdots,\quad \bar{\Phi}(\mu,s),\quad \cdots\right]^\mathrm{T}, \tag{13}$$

$$\bar{\boldsymbol{F}}_\mathrm{e}(s) = \left[\cdots,\quad \bar{F}_\mathrm{e}(\mu,s),\quad \cdots\right]^\mathrm{T}, \tag{14}$$

and the state matrix is given as

$$\boldsymbol{\mathcal{A}} = \mathrm{diag}\left(\ldots, s_\mu, \ldots\right). \tag{15}$$

3.2. Output Equation

The inverse transformation from Eq. (4) can be reformulated in vector form as an output equation for the state-space model

$$\boldsymbol{Y}(x,s) = \boldsymbol{\mathcal{C}}(x)\bar{\boldsymbol{Y}}(s), \tag{16}$$

with the matrix

$$\boldsymbol{\mathcal{C}}(x) = \left[\ldots,\ \frac{1}{N_\mu}\boldsymbol{K}(x,\mu),\ \ldots\right]. \tag{17}$$

3.3. Transformed Boundary Term

The transformed boundary term $\bar{\Phi}$ in Eq. (6), can be reformulated in terms of a vector notation to fit into the state equation (11)

$$\bar{\boldsymbol{\Phi}}(s) = \boldsymbol{\mathcal{B}}(0)\boldsymbol{\Phi}(0,s) - \boldsymbol{\mathcal{B}}(\ell)\boldsymbol{\Phi}(\ell,s), \tag{18}$$

with the matrix

$$\boldsymbol{\mathcal{B}}(x) = \begin{bmatrix} \vdots \\ \tilde{\boldsymbol{K}}^\mathrm{H}(x,\mu) \\ \vdots \end{bmatrix}, \qquad x=0,\ell. \tag{19}$$

The state space description with matrices of possibly infinite size is only used to show the parallelism to other state-space descriptions. The matrices $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}(x)$ and $\boldsymbol{\mathcal{C}}(x)$ act as transformation operators rather than as matrices in the sense of linear algebra.

4. INPUT/OUTPUT BEHAVIOUR

When dealing with boundary conditions of physical systems it is important to select which quantities are defined as input and which are ouputs of the system (see [10, p. 55]). The number of boundary conditions corresponds to the order of the spatial differential operator, respectively the number of variables in the vector $\boldsymbol{Y}(x,s)$. In many practical cases, boundary conditions are assigned to half the variables at $x=0$ and to half of the variables at $x=\ell$. The other variables at either side depend on the system dynamics and can be observed, they are called boundary observations here.

With this distribution of boundary conditions and boundary observations, the vector of variables $\boldsymbol{Y}$ can be sorted in a way, that the boundary conditions appear on the top of the vector $\boldsymbol{Y}(x,s)$ as $\boldsymbol{Y}_1(x,s)$ and those variables, which are assigned to boundary observations appear on the bottom as $\boldsymbol{Y}_2(x,s)$. To construct these vectors, the variable vector is multiplied by suitable permutation matrices to extract the relevant entries, which leads to a separated representation of the vector of variables

$$\boldsymbol{Y}(x,s) = \begin{bmatrix} \boldsymbol{Y}_1(x,s) \\ \boldsymbol{Y}_2(x,s) \end{bmatrix}. \tag{20}$$

This partitioning carries over to the whole state space description. The eigenfunctions $\boldsymbol{K}$ and $\tilde{\boldsymbol{K}}$ from (3), (4) have the same size as the vector $\boldsymbol{Y}(x,s)$, they can also be divided by applying the same permutation

$$\boldsymbol{K}(x,\mu) = \begin{bmatrix} \boldsymbol{K}_1(x,\mu) \\ \boldsymbol{K}_2(x,\mu) \end{bmatrix}, \quad \tilde{\boldsymbol{K}}(x,\mu) = \begin{bmatrix} \tilde{\boldsymbol{K}}_1(x,\mu) \\ \tilde{\boldsymbol{K}}_2(x,\mu) \end{bmatrix}. \tag{21}$$

With this partitioning, the matrices in the output equation (16) and the transformed boundary vector (18) are also divided

$$\boldsymbol{\mathcal{C}}(x) = \begin{bmatrix} \boldsymbol{\mathcal{C}}_1(x) \\ \boldsymbol{\mathcal{C}}_2(x) \end{bmatrix}, \qquad \boldsymbol{\mathcal{B}}(x) = \begin{bmatrix} \boldsymbol{\mathcal{B}}_1(x) & \boldsymbol{\mathcal{B}}_2(x) \end{bmatrix}. \tag{22}$$

4.1. Boundary Conditions

After selecting the input and output variables, the following sections discuss different kinds of boundary conditions and their connection to each other. Sec. 6 presents an application of these general concepts to a physical model of a vibrating string.

4.1.1. Simple Boundary Conditions

This section defines simple boundary conditions at $x=0,\ell$ of the physical system. Simple boundary conditions are conditions acting on individual entries of $\boldsymbol{Y}_1(x,s)$ at the outputs of the system. At first the boundary matrices and vectors according to (2) are specialized to

$$\boldsymbol{F}_\mathrm{b}(x,s) = \boldsymbol{F}_\mathrm{s}(x), \qquad \boldsymbol{\Phi}(x,s) = \boldsymbol{\Phi}_\mathrm{s}(x,s). \tag{23}$$

with

$$\boldsymbol{F}_\mathrm{s}(x) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{\Phi}_\mathrm{s}(x,s) = \begin{bmatrix} \boldsymbol{\Phi}_\mathrm{s1}(x,s) \\ \boldsymbol{0} \end{bmatrix}. \tag{24}$$

Additionally a matrix of boundary observations is defined

$$\boldsymbol{G}_\mathrm{s}^\mathrm{H}(x) = \boldsymbol{I} - \boldsymbol{F}_\mathrm{s}^\mathrm{H}(x) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{F}_\mathrm{s}^\mathrm{H}(x) + \boldsymbol{G}_\mathrm{s}^\mathrm{H}(x) = \boldsymbol{I}. \tag{25}$$

With these two matrices the input and output behaviour of the system is described completely. Applying the matrices of boundary conditions (23) and boundary observations (25) to the vector of variables (22) leads to (some arguments (x,s) omitted)

$$\boldsymbol{F}_\mathrm{s}^\mathrm{H}\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{\Phi}_\mathrm{s1} \\ \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{G}_\mathrm{s}^\mathrm{H}\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{Y}_\mathrm{so} \end{bmatrix}, \tag{26}$$

$$\boldsymbol{Y}_1(x,s) = \boldsymbol{\Phi}_\mathrm{s1}(x,s), \qquad \boldsymbol{Y}_2(x,s) = \boldsymbol{Y}_\mathrm{so}(x,s). \tag{27}$$

4.1.2. Complex Boundary Conditions

Complex boundary conditions are defined here as linear combinations of physical quantities at the system boundaries $x=0,\ell$, including time derivatives or complex parameters and functions. Like simple boundary conditions, also complex boundary conditions can be defined in a more general form. Again the matrix of boundary conditions is specialized

$$\boldsymbol{F}_\mathrm{b}(x,s) = \boldsymbol{F}_\mathrm{c}(x,s), \qquad \boldsymbol{\Phi}(x,s) = \boldsymbol{\Phi}_\mathrm{c}(x,s). \tag{28}$$

with

$$\boldsymbol{F}_\mathrm{c}(x,s) = \begin{bmatrix} \boldsymbol{F}_\mathrm{c1}(x,s) & 0 \\ \boldsymbol{F}_\mathrm{c2}(x,s) & 0 \end{bmatrix}, \qquad \boldsymbol{\Phi}_\mathrm{c} = \begin{bmatrix} \boldsymbol{\Phi}_\mathrm{c1}(x,s) \\ \boldsymbol{0} \end{bmatrix}. \tag{29}$$

The matrix $\boldsymbol{F}_\mathrm{c1}(x,s)$ has to be chosen as non-singular, except for isolated values of s. Additionally, as in the case of simple boundary conditions, a matrix of boundary observations is defined

$$\boldsymbol{G}_\mathrm{c}(x,s) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{G}_\mathrm{c1}(x,s) \\ \boldsymbol{0} & \boldsymbol{G}_\mathrm{c2}(x,s) \end{bmatrix}, \tag{30}$$

which transforms the vector of variables into a vector of complex boundary observations

$$\boldsymbol{G}_\mathrm{c}^\mathrm{H}(x,s)\boldsymbol{Y}(x,s) = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{Y}_\mathrm{co}(x,s) \end{bmatrix}. \tag{31}$$

Also with the complex boundary conditions the input and output behaviour of the system can be described in a closed form, by applying both matrices (29), (30) to the vector of variables, which leads to

$$\boldsymbol{F}_\mathrm{c1}^\mathrm{H}(x,s)\boldsymbol{Y}_1(x,s) + \boldsymbol{F}_\mathrm{c2}^\mathrm{H}(x,s)\boldsymbol{Y}_2(x,s) = \boldsymbol{\Phi}_\mathrm{c1}(x,s), \tag{32}$$

$$\boldsymbol{G}_\mathrm{c1}^\mathrm{H}(x,s)\boldsymbol{Y}_1(x,s) + \boldsymbol{G}_\mathrm{c2}^\mathrm{H}(x,s)\boldsymbol{Y}_2(x,s) = \boldsymbol{Y}_\mathrm{co}(x,s). \tag{33}$$

4.2. Connection between simple and complex Conditions

With the definitions of simple and complex boundary conditions from Sec. 4.1.1 – 4.1.2 their connections can be explored, to express complex boundary conditions in terms of the simple ones. So in general the physical system is described by a bi-orthogonal basis for simple boundary conditions. Then the external connections are formulated in terms of relations between the input and output variables of the complex boundary conditions.

4.2.1. Simple and Complex Boundary Conditions

Starting with the application of complex boundary conditions to the vector of variables $\boldsymbol{Y}$ and exploiting Eq. (25) leads to

$$\boldsymbol{F}_\mathrm{c}^\mathrm{H}(x,s)\left(\boldsymbol{F}_\mathrm{s}^\mathrm{H}(x) + \boldsymbol{G}_\mathrm{s}^\mathrm{H}(x)\right)\boldsymbol{Y}(x,s) = \boldsymbol{\Phi}_\mathrm{c}(x,s). \tag{34}$$

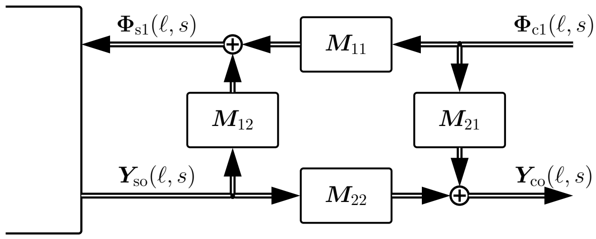

Figure 1: *Boundary circuit for the general connection between simple and complex boundary conditions according to Eq.* (35)*,* (38)*. $\Phi_{s1}(\ell,s), Y_{so}(\ell,s)$: Inputs and observations of the model for simple boundary conditions. $\Phi_{c1}(\ell,s), Y_{co}(\ell,s)$: Inputs and observations of the complex boundary circuit. M: Matrices defining the connection between simple and complex boundary conditions.*

Inserting the matrix of complex boundary conditions (29) and the definitions of matrices of the simple case (24), (25) and applying the block partitioning leads to an expression of the simple boundary conditions Φ_{s1} in terms of the complex boundary conditions Φ_{c1} and simple boundary observations Y_{so}

$$\Phi_{s1}(x,s) = M_{11}(x,s)\Phi_{c1}(x,s) + M_{12}(x,s)Y_{so}, \tag{35}$$

with the connection matrices

$$M_{11}(x,s) = F_{c1}^{-H}(x,s), \tag{36}$$
$$M_{12}(x,s) = -F_{c1}^{-H}(x,s)F_{c2}^{H}(x,s). \tag{37}$$

4.2.2. Simple and Complex Boundary Observations

Similar to the boundary conditions, also the boundary observation of the simple and complex case can be connected. The goal is to express the complex boundary observations Y_{co} in terms of the simple boundary values Φ_{s1} and the simple boundary observations Y_{so}. Starting with Eq. (33) and exploiting the block structure of matrices and vectors, together with (27), leads to

$$Y_{co}(x,s) = M_{21}(x,s)\Phi_{c1}(x,s) + M_{22}Y_{so}(x,s), \tag{38}$$

with the connection matrices

$$M_{21}(x,s) = G_{c1}^{H}(x,s)F_{c1}^{-H}(x,s), \tag{39}$$
$$M_{22}(x,s) = G_{c2}^{H}(x,s) - G_{c1}^{H}(x,s)F_{c1}^{-H}(x,s)F_{c2}^{H}(x,s). \tag{40}$$

Eq. (35) and (38) describe the connection between simple and complex boundary conditions in terms of boundary values and boundary observations.

Fig. 1 shows this connection between simple and complex boundary conditions for a system boundary (open box on the left side) at $x = \ell$. The physical system on the left side is designed with simple boundary conditions, so that the behaviour at $x = \ell$ is defined by the partitioned vector Y from (20), respectively by the simple boundary values Φ_{s1} and the simple boundary observations Y_{so} according to Eq. (27). The figure shows that whenever a system with complex boundary conditions is desired, it is not necessary to redesign the whole physical model. Instead Eqs. (35) and (38) allow to impose the complex boundary values Φ_{c1} on the simple model and to express the complex boundary observations Y_{co} in terms of the simple ones.

5. MODIFIED STATE SPACE DESCRIPTION

In this section the connection between simple and complex boundary conditions from Sec. 4.2 is incorporated into the state-space description from Sec. 3.

5.1. Transformed Boundary Term

The simple boundary observations Y_{so} can be derived by the application of the partitioning (20), (22) to the output equation (16)

$$Y_{so}(x,s) = \mathcal{C}_2(x)\bar{Y}(s), \qquad x = 0,\ell. \tag{41}$$

In the same way, the partitioning (20), (22) together with the simple boundary conditions from Eq. (24) is applied to the transformed boundary term from Eq. (18). The result is

$$\bar{\Phi}(s) = \mathcal{B}_1(0)\Phi_{s1}(0,s) - \mathcal{B}_1(\ell)\Phi_{s1}(\ell,s) = \hat{\mathcal{B}}\hat{\Phi}(s), \tag{42}$$

with the block matrices

$$\hat{\mathcal{B}} = \begin{bmatrix} \mathcal{B}_1(0) \\ -\mathcal{B}_1(\ell) \end{bmatrix}^{T}, \qquad \hat{\Phi}_s(s) = \begin{bmatrix} \Phi_{s1}(0,s) \\ \Phi_{s1}(\ell,s) \end{bmatrix}. \tag{43}$$

Now the connection between simple and complex boundary conditions can be incorporated into the transformed boundary term by inserting Eq. (35) with the boundary observations from (41) into (42). Solving for $\bar{\Phi}(s)$ leads to a representation of the boundary term $\bar{\Phi}$ by the state variable $\bar{Y}(s)$ and the complex boundary conditions

$$\bar{\Phi}(s) = -\hat{\mathcal{B}}\hat{\mathcal{K}}\bar{Y}(s) + \mathcal{B}_c\hat{\Phi}_c(s), \tag{44}$$

with the block matrices

$$\hat{\mathcal{K}} = -\begin{bmatrix} M_{12}(0,s)\mathcal{C}_2(0) \\ M_{12}(\ell,s)\mathcal{C}_2(\ell) \end{bmatrix}, \quad \hat{\Phi}_c(s) = \begin{bmatrix} \Phi_{c1}(0,s) \\ \Phi_{c1}(\ell,s) \end{bmatrix}, \tag{45}$$

$$\mathcal{B}_c = \begin{bmatrix} \mathcal{B}_1(0)M_{11}(0,s) & -\mathcal{B}_1(\ell)M_{11}(\ell,s) \end{bmatrix}. \tag{46}$$

5.2. State Equation and Output Equation

Including (44) into the state equation (11) leads to a modified state equation, where the complex boundary conditions are incorporated

$$s\bar{Y}(s) = \mathcal{A}_c\bar{Y}(s) + \mathcal{B}_c\hat{\Phi}_c(s) + \bar{F}_e(s), \tag{47}$$

with the modified state feedback matrix

$$\mathcal{A}_c = \mathcal{A} - \hat{\mathcal{B}}\hat{\mathcal{K}}. \tag{48}$$

The structure of the modified state equation shows that the boundary circuit from Fig. 1 is equivalent to a state feedback structure. The state matrix $\mathcal{A}$ of the state space description contains the poles of the physical system, Eq. (48) shows the influence of the boundary circuit which shifts the poles of the system with simple boundary conditions.

The output equation (16) of the state-space description is not directly affected by the incorporation of complex boundary conditions.

5.3. Relation to Control Theory

The modification of the open loop behaviour by feedback as introduced in Sec. 5.2 is well known in the literature on control theory [8,10]. The fact that system interconnection constitutes a feedback control loop is highlighted e.g. in [10, Fig. 15]. The notation in (48) has been chosen to reflect the description of closed loop systems in e.g. [9, Eq. (19)].

6. STRING MODEL

This section presents a multidimensional transfer function model of a guitar string based on [7]. The derivations are omitted for brevity; instead the results and especially the boundary conditions from [7] are rewritten to fit the input/output model from Sec. 4.

6.1. Physical Description

A single vibrating string can be described by a PDE [1, Eq. (6)] in terms of the deflection $y = y(x,t)$ depending on time t and space x on the string

$$\rho A\ddot{y} + EIy'''' - T_\mathrm{s} y'' + d_1\dot{y} - d_3\dot{y}'' = f_\mathrm{e}, \tag{49}$$

where $\dot{y}$ represents the time- and y' the space-derivative and with the cross-section area A, moment of inertia I and the length ℓ. The material is characterized by the density ρ and Young's modulus E. T_s describes the tension and d_1 and d_3 are frequency independent and frequency dependent damping [1]. The space and time dependent excitation function is defined as $f_\mathrm{e} = f_\mathrm{e}(x,t)$.

The PDE in (49) can be rearranged in the vector form (1) as shown in [7]. To fit the vector of variables into the input/output model (20), the vector is rearranged with suitable permutation matrices. Therefore the vector of variables defined in [7] can be partitioned

$$\mathbf{Y}_1(x,s) = \begin{bmatrix} sY(x,s) \\ Y''(x,s) \end{bmatrix}, \quad \mathbf{Y}_2(x,s) = \begin{bmatrix} Y'(x,s) \\ Y'''(x,s) \end{bmatrix}. \tag{50}$$

6.2. Simple Boundary Conditions

A set of simple boundary conditions for a string fixed at both ends $x = 0, \ell$ is formulated as a set of equations for the deflection and bending moment

$$sY(x,s) = \Phi_1(x,s), \quad Y''(x,s) = \Phi_2(x,s) \quad x = 0, \ell. \tag{51}$$

This set of linear equations can be formulated in terms of a matrix of boundary conditions and a vector of boundary excitations according to (2) as shown in [7]. From here on boundary conditions of the first kind are called simple boundary conditions and they are formulated as described in Sec. 4.1.1 in Eq. (24) with

$$\mathbf{\Phi}_\mathrm{s1}(x,s) = \begin{bmatrix} \Phi_1(x,s) \\ \Phi_2(x,s) \end{bmatrix}, \qquad x = 0, \ell. \tag{52}$$

6.3. Complex Boundary Conditions

As a set of complex boundary conditions, impedance boundary conditions are used at $x = 0$ according to [7]. They are formulated as a set of linear equations, combining the string deflection Y with a force F by a frequency dependent admittance $Y_\mathrm{b}(s)$

$$sY(0,s) - Y_\mathrm{b}(s)F(0,s) = \Phi_\mathrm{c1}, \quad Y''(0,s) = \Phi_\mathrm{c2}. \tag{53}$$

The force can be expressed as a combination of two forces reformulated in terms of the derivatives of string deflection [13]

$$F(0,s) = T_\mathrm{s} Y'(0,s) - EIY'''(0,s). \tag{54}$$

The boundary conditions are rewritten as shown in Sec. 4.1.2 to fit into the input/output model, with the matrices of boundary conditions

$$\boldsymbol{F}^\mathrm{H}_\mathrm{c1}(0,s) = \boldsymbol{I}, \quad \boldsymbol{F}^\mathrm{H}_\mathrm{c2}(0,s) = Y_\mathrm{b}(s)\begin{bmatrix} -T_\mathrm{s} & EI \\ 0 & 0 \end{bmatrix}, \tag{55}$$

and the vector of boundary excitations

$$\mathbf{\Phi}_\mathrm{c1}(0,s) = \begin{bmatrix} \Phi_\mathrm{c1}(0,s) \\ \Phi_\mathrm{c2}(0,s) \end{bmatrix}. \tag{56}$$

For the position $x = \ell$ the simple boundary conditions from Sec. 6.2 are applied. It then follows for the matrix of boundary conditions at $x = \ell$

$$\boldsymbol{F}^\mathrm{H}_\mathrm{c1}(\ell,s) = \boldsymbol{I}, \qquad \boldsymbol{F}^\mathrm{H}_\mathrm{c2}(\ell,s) = \mathbf{0}, \tag{57}$$

and for the boundary observations

$$\mathbf{\Phi}_\mathrm{c1}(\ell,s) = \mathbf{\Phi}_\mathrm{s1}(\ell,s). \tag{58}$$

6.4. Kernel Functions

To fit into the input/output model from Sec. 4 also the kernel functions from [7] have to be rearranged. For the primal kernel function follows according to (21) with suitable permutations

$$\boldsymbol{K}_1(x,\mu) = \begin{bmatrix} \frac{s_\mu}{\gamma_\mu}\sin(\gamma_\mu x) \\ -\gamma_\mu \sin(\gamma_\mu x) \end{bmatrix}, \quad \boldsymbol{K}_2(x,\mu) = \begin{bmatrix} \cos(\gamma_\mu x) \\ -\gamma_\mu^2 \cos(\gamma_\mu x) \end{bmatrix}, \tag{59}$$

where $\boldsymbol{K}_1$ is assigned to the boundary conditions and $\boldsymbol{K}_2$ to the boundary observations. The same partitioning is applied to the adjoint kernel function

$$\tilde{\boldsymbol{K}}_1(x,\mu) = \begin{bmatrix} q_1^* \cos(\gamma_\mu x) \\ -\gamma_\mu^2 \cos(\gamma_\mu x) \end{bmatrix}, \quad \tilde{\boldsymbol{K}}_2(x,\mu) = \begin{bmatrix} -\frac{s_\mu^* q_1^*}{\gamma_\mu}\sin(\gamma_\mu x) \\ \gamma_\mu \sin(\gamma_\mu x) \end{bmatrix}, \tag{60}$$

with the wave numbers γ_μ and the coefficient q_1 defined in [7].

6.5. Modified State Space Model

For the synthesis of the string model (49) normally Eq. (4) can be used as a superposition of first order systems. Alternatively a state-space model can be set up for the synthesis (see Sec. 3). Here the modified state-space model from Sec. 5 is applied for the incorporation of complex boundary conditions (see Sec. 6.3).

6.5.1. Output Equation

The output equation is designed according to Eq. (16) with the output matrix $\boldsymbol{\mathcal{C}}(x)$ from Eq. (17), which is divided according to (22). For the design of the matrices $\boldsymbol{\mathcal{C}}_1$ and $\boldsymbol{\mathcal{C}}_2$, the kernel functions (59), (60) and the scaling factor N_μ (see [7, Eq. (15)]) are necessary.

6.5.2. State Equation

The state equation for the synthesis of the string model is designed according to Eq. (47). The modified state feedback matrix (48) is constructed via the state matrix $\boldsymbol{\mathcal{A}}$ using the eigenfrequencies as the solution of the dispersion relation [7]

$$s_\mu^2 + \left(\frac{a_1}{c_1} - \lambda^2 \frac{c_2}{c_1}\right) s_\mu - \frac{\lambda^2}{c_1}\left(\lambda^2 + a_2\right) = 0. \tag{61}$$

The extension of the state matrix to incorporate complex boundary conditions follows the principle shown in Sec. 5 using the matrices in Eq. (45), (46).
These matrices are established with

- the feedback matrices $\boldsymbol{M}_{12}$ and $\boldsymbol{M}_{11}$ from (36)-(37) with the complex boundary conditions from (55), (57),
- the partitioned output matrices $\boldsymbol{\mathcal{C}}(x)$ according to (22) and (17),
- the complex boundary excitations (56), (58),
- the transformation matrices of the transformed boundary term $\bar{\boldsymbol{\Phi}}$ according to (42) and (19).

With a state equation designed in this form, the complex boundary conditions are incorporated into a string model designed with simple boundary conditions as shown in Fig. 1.

7. BRIDGE MODEL

As a realistic boundary circuit for the impedance boundary conditions (53), a frequency dependent bridge model is chosen as discussed in [6]. A bridge model resembles the influence of a guitar bridge, where the string is attached to the bridge at the position $x = 0$.

For simplicity the bridge admittance is designed as the superposition of N second order bandpass filters

$$Y_\mathrm{b}(s) = \sum_{n=1}^{N} Y_n \frac{s\frac{\omega_0}{Q_n}}{s^2 + \frac{\omega_0}{Q_n}s + \omega_0^2}, \tag{62}$$

with the quality factor Q_n and the resonance frequency $\omega_n = 2\pi f_n$ for each mode. An additional gain factor Y_n is added to each bandpass. In general the admittance functions for the bridge are position and direction dependent, so the admittance differs for different strings [6].

The physical parameters for the bridge resonances can be taken from measurements as e.g. shown in [14]. There the lowest six most significant eigenmodes of the bridge are realized with damped sinusoids.

Figure 2 shows the first six prominent modes of the bridge admittance $Y_\mathrm{b}(s)$ in the frequency domain. The resonance frequencies ω_n and the quality factors Q_n are taken from [14]. The gain factors are related to the effective masses $Y_n = \frac{1}{m_n}$ as presented in [6], with the physical values for the effective masses from [14].

For simulations in the continuous frequency domain, the admittance function from Eq. (62) is directly used. The function is just inserted into the boundary conditions (55) and incorporated into the modified state space model described in Sec. 6.5.

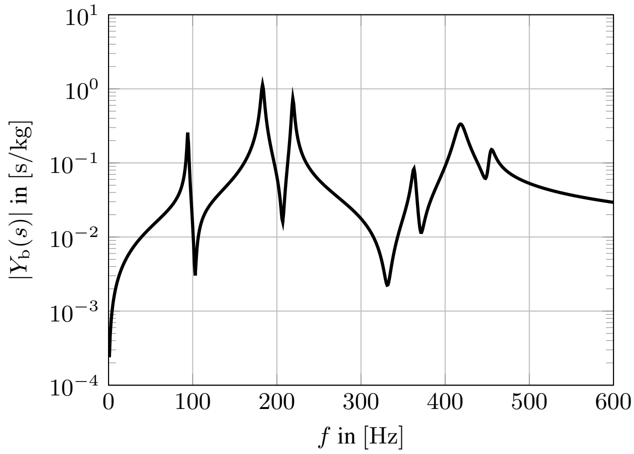

Figure 2: *Absolute value of the bridge admittance $Y_\mathrm{b}(s)$ according to Eq.* (62)*. Shown are the $N = 6$ prominent modes [14].*

8. SIMULATION RESULTS

The following section shows the simulation results for a guitar string including a boundary circuit. All simulation results are computed using the modified state-space model as shown in Sec. 6.5, with the state equation (47) and the output equation (16).

All calculations are performed in the continuous frequency domain and the results are presented in terms of the amplitude spectra of the bending force $F_\mathrm{bend}(x, s)$ in y-direction in the lower frequency range. The bending force is calculated according to

$$F_\mathrm{bend}(x, s) = EIY'''(x, s), \tag{63}$$

where several valid simplifications were applied [3, 13].

8.1. Boundary Conditions and Excitation Function

The complex boundary conditions are chosen for the frequency domain simulations of the string according to Sec. 6.3. The boundary values $\boldsymbol{\Phi}_{\mathrm{c}1}, \boldsymbol{\Phi}_{\mathrm{c}2}$ of the complex boundary conditions are set to zero for brevity. The admittance $Y_\mathrm{b}(s)$ is interpreted as the admittance of the guitar bridge, where the string is placed on. The admittance is varied for the following simulations.

The function $f_\mathrm{e}(x, t)$ exciting the string is a single impulse at the position $x_\mathrm{e} = 0.3\,\mathrm{m}$ on the string, as shown in [7]. In the frequency domain this excitation leads to a uniform excitation of all frequency components. The simulation of output variables in response to the excitation function can be seen as an impedance-analysis of the string.

For all simulations a nylon guitar low E-String is used. The values for the physical parameters in Eq.(49) are taken from [6, 15, 16].

8.2. Frequency independent Bridge Impedance

In a first step the simulation is performed for a frequency independent constant bridge admittance $Y_\mathrm{b}(s) = \mathrm{const.}$ Figure 3 shows the normalized amplitude spectra of the bending force $|F_\mathrm{bend}(0, s)|$ at the bridge position $x = 0$ for the lower frequency range. The bridge admittance is varied $Y_\mathrm{b}(s) = 0\,\mathrm{s/kg}, 2\,\mathrm{s/kg}, 5\,\mathrm{s/kg}$.

For zero admittance (solid curve in Fig. 3) the modes of the guitar string can be seen clearly at the fundamental frequency and the higher modes. The behaviour complies with that of a string model with simple boundary conditions as the feedback path in the modified state matrix (48) is zero. The results for a string with simple boundary conditions are confirmed in [5].

For increasing bridge admittance (dotted, dashed curve in Fig. 3) all modes of the guitar string are damped equally. These results are not realistic as a real bridge impedance is strongly frequency dependent. The results for a frequency independent admittance show the validity of the presented concepts for a feedback structure for a generic example and the results of [7] are verified.

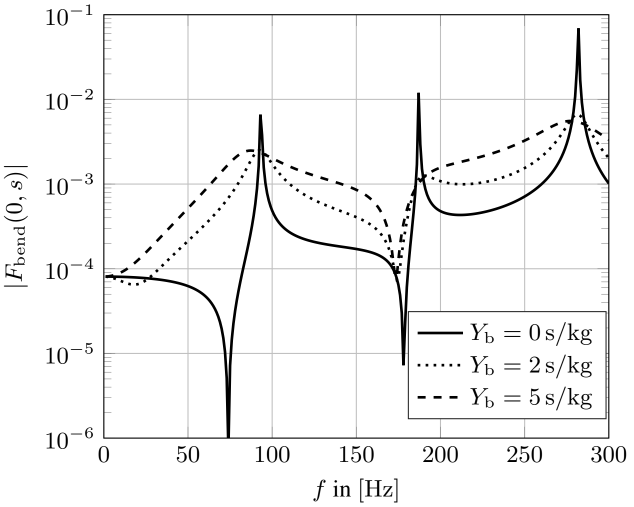

Figure 3: *Absolute value of the spectrum of bending force $F_{\mathrm{bend}}(0, s)$ at the bridge position $x = 0$, for three different frequency independent bridge admittances $Y_{\mathrm{b}} = \mathrm{const}$.*

8.3. Frequency dependent Bridge Impedance

Now a frequency dependent bridge admittance $Y_{\mathrm{b}}(s)$ is used for the simulations according to Sec. 7 including the first six ($N = 1, \ldots, 6$) modes of a measured bridge impedance (see Fig. 2). The physical values for the mode resonance frequencies f_n, the Q-values and the effective masses are taken from [14, Table 1].

The results of the string simulation for a frequency dependent bridge model are pictured in Fig. 4. The figure shows the amplitude spectrum of bending force $F_{\mathrm{bend}}(0, s)$ at $x = 0$ for a zero bridge admittance $Y_{\mathrm{b}} = 0$ (solid curve) and a frequency dependent bridge admittance $Y_{\mathrm{b}}(s)$ according to Eq. (62) (dotted curve).

To have a comprehensive representation of the string behaviour the bridge admittance (as shown in Fig. 2) is plotted into Fig. 4 to indicate the mode positions of the bridge model (dashed curve). The frequency range is limited to the interesting range, where the bridge modes influence the string vibration [14, Table 1].

In Fig. 4 the influence of the bridge admittance on the string vibration can be seen clearly. Depending on the resonance frequencies f_n and the corresponding amplitudes of the bridge admittance, the modes of the vibrating string are shaped. Some of the peaks are damped completely (around 180 Hz in Fig. 4) other peaks are shifted in frequency according to the profile of $Y_{\mathrm{b}}(s)$.

Fig. 5 shows the influence of the bridge admittance $Y_{\mathrm{b}}(s)$ on the bending force $F_{\mathrm{bend}}(x_{\mathrm{e}}, s)$ according to Eq. (63) at the excitation position $x = x_{\mathrm{e}}$. It can be seen that in the frequency independent case (dotted line) all string resonances are damped in the same way. For a frequency dependent bridge admittance (dashed line) the modes are influenced according to the shape of the bridge admittance.

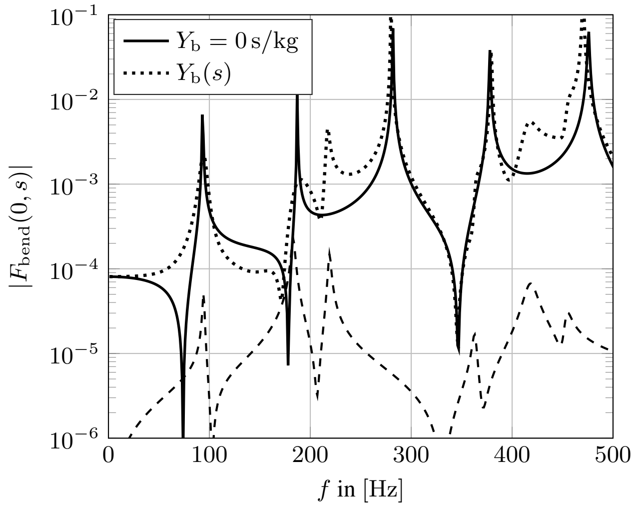

Figure 4: *Absolute value of the spectrum of bending force $F_{\mathrm{bend}}(0, s)$ at the bridge position $x = 0$ for zero bridge admittance (solid line) and for a frequency dependent bridge admittance according to Eq. (62) in Sec. 7 (dotted line). The dashed black line is an amplitude shifted version of the bridge admittance $Y_{\mathrm{b}}(s)$, plotted for illustration.*

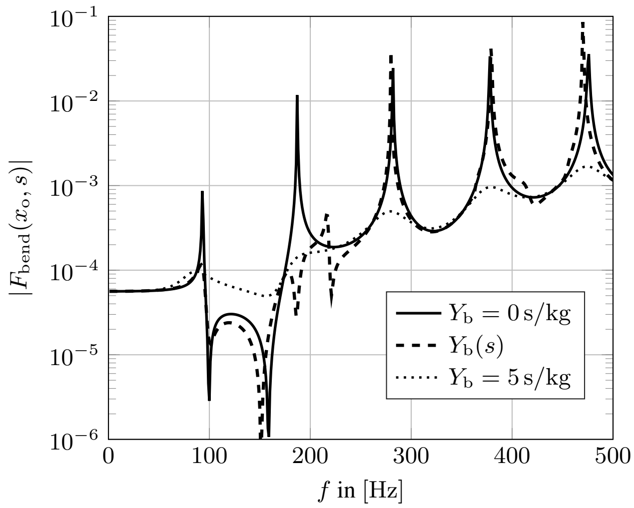

Figure 5: *Absolute value of the spectrum of bending force $F_{\mathrm{bend}}(x_{\mathrm{e}}, s)$ at the excitation position $x_{\mathrm{e}} = 0.3\,\mathrm{m}$ for zero bridge admittance (solid line), for a frequency dependent bridge admittance according to Eq. (62) in Sec. 7 (dashed line) and for a constant frequency independent bridge admittance $Y_{\mathrm{b}} = \mathrm{const}$ (dotted line).*

9. FURTHER WORKS

This contribution presents a general approach for the incorporation of complex boundary conditions into systems designed with simple boundary conditions based on concepts from control theory [8–10]. This method can be used in many applications and can be further improved. Therefore the following further works are envisaged: The bridge admittance (62) was chosen exemplary as a superposition of bandpass filters including realistic resonance frequencies and quality factors based on [14]. For further works the model can be extended to include more realistic damping effects. The concepts shown here can be used to link a body model to the bridge in a block-based style.

The presented concepts are based on a modified state-space description (see Sec. 5) including a feedback matrix in the continuous frequency domain. These concepts can be transformed into the discrete-time domain for real-time simulations, which requires a re-design of the state-space description from Sec. 3 and Sec. 5. Suitable state space techniques have been presented in [7, Secs. 3-5].

Interactions of the model at any position on the string also can be realized with the presented concepts. According to Sec. 4 an input/output model including the excitation function $f_{\mathrm{e}}(x,t)$ can be derived, e.g. for string-fretboard or string-finger interaction.

The concept for the incorporation of complex boundary conditions can be extended to more than one spatial dimension. E.g. the eigenfunctions for the 2D plate equations cannot be derived analytically except for the most simple boundary conditions [17]. The presented concept appears to be a promising approach for the simulation of the plate equation with complex boundary conditions.

10. CONCLUSIONS

This paper presented a concept for the incorporation of complex boundary conditions into systems designed with simple boundary conditions. Building on prior work this contribution develops an input/output description for systems based on transfer function models. Subsequently it has been shown that complex boundary conditions (e.g. impedance boundary conditions) can be included into a model by the design of a feedback loop related to control theory. The concepts allow to change the boundary behaviour of a system without changing the interior model of the system.

The validity of the presented concepts is verified by an extensive example for modelling of guitar strings. A bridge model is added to an existing string model based on a multidimensional transfer function. The results are presented in terms of several spectra of the string resonances.

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